

# Dynamics in Art of War \*

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## Abstract

This paper examines basic principles in Sun Tzu's classic treatise Art of War. In a dynamic decision-theoretic model, there is a potential conflict between two sides. The comparative statics results show precise conditions under which the principles of strategic fighting in Art of War hold.

**Keywords:** Confrontation, Time Preference, Stopping Game, Best Response, Patience

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“The good fighters of old first put themselves beyond the possibility of defeat, and then waited for an opportunity of defeating the enemy.” Sun Tzu, Art of War.

## 1 Introduction

Some books are so valuable that generations have struggled for its preservation and understanding. The canonical example is Sun Tzu’s military treatise Art of War Tzu (1961, original traditionally dated circa 500 BC). Seemingly impervious to time, Art of War is perhaps the most famous study of strategy ever written. It inspires people today on matters of business, personal conduct and even romance, as it once guided kings, generals and strategists millennia ago on armed conflict.

There are two main keys to the success of Art of War (in the last 15 years, more than 20 popular books apply the principles of Art of War to business practice, romance, self-improvement, and several other matters).<sup>1</sup> First, it works out the basic principles of strategic fighting (e.g., when to be aggressive or defensive depending on circumstances). These principles proved to be practical and revolutionary. They made clear that conflict is a more subtle and complex matter than the traditional concept of war of attrition where one side simply uses greater material resources to wear out and overpower the other side. The second main key is the subject itself: Conflict is common in ordinary life and confrontations, no matter how disguised, have always occurred (e.g., divorce, commercial litigation, professional disputes). Hence, there is a perennial interest in the idea of strategic fighting.

The ideas in Art of War, while still relevant in modern times, have not been examined by formal models. We revisit parts of ancient eastern literature under modern mathematical lenses. Our focus is on a central theme in Art of War: the idea that a confrontation is a serious affair which must not be initiated on impulse and without careful planning. If started, it must be for strategic reasons. Thus, a basic question in Art of War is when to engage in a confrontation: “One who knows when he can fight and when he cannot fight, will be victorious.” To properly address this question, we must consider a setting where the timing of the confrontation is critical for the

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<sup>1</sup> Wee (1991) notes that Art of War is a required reading in Japanese business schools. For applications of Art of War in various aspects modern life see also McNeilly and McNeilly (2011), Sheetz-Runkle (2014), Smith (2008), Rogell (2010), Bell (2009).

resolution of the conflict and is taken strategically.

We examine a decision-theoretic, infinite horizon model where, at every period, each side decides whether to engage the other side in a confrontation. If anyone attacks, the process ends with one side as the victor. Otherwise, they both live to face each other the next day. The odds of victory changes every period and are known to both sides before deciding whether to strike.

Sun Tzu recommends a defensive position while waiting for an opportune moment to attack: “The good fighters of old first put themselves beyond the possibility of defeat, and then waited for an opportunity of defeating the enemy.” This tenet, combined with many examples of catastrophic outcomes produced by hasty engagements, suggest that a more patient individual could wait longer for a proper opportunity to strike. So, there should be a negative relation between patience and aggressiveness (i.e., the propensity to fight sooner rather than later). This may seem commonsensical. An appeal to patience is normally understood as an appeal to serenity and seldom, if ever, to aggressive action. But here, as in many of Sun Tzu’s aphorisms, a point is concealed. Forgoing an opportunity to attack does indeed give an option value of a better opportunity to attack, but it can also be fatal because it also gives the enemy the option value of a better opportunity to attack. Thus, in conflicts, the results of patience are unclear.

Our results are as follows: If the probability of victory is always high (i.e., above a threshold), then a more patient individual requires higher odds of success to attack and so, as Sun Tzu points out, waits longer to start a conflict. This holds under Sun Tzu’s requirement of a position where defeat is never likely. If this proviso does not hold and likelihood of victory is always low then the relationship between patience and aggressiveness (i.e., the tendency to attack sooner) is reversed. A more patient individual requires lower odds of success to attack. Under vulnerable conditions, patient individuals may preempt the opponent with early strikes.

In the case that no side has, ex-ante, an upper hand in the conflict then the relationship between patience and aggressiveness depends on the strategy of the opponent. If the opponent is not aggressive (i.e., attacks only when the odds of victory are above a high threshold) then a more patient individual is less aggressive. Conversely, if the opponent is aggressive then a more patient individual is more aggressive. Aggressiveness is best responded with aggressiveness. If the opponent is sufficiently aggressive then a patient individual attacks even if the current expected payoff of confronting the opponent is negative.

These results formalize a new way to validate Sun Tzu's maxims. A far-sighted commander must forgo an opportunity to attack that a myopic commander would take, and wait for a better opportunity to strike. However, *this is only so under Sun Tzu's proviso of a strong defensive position*. Under vulnerable conditions, the relationship between patience and aggressiveness is reversed. A far-sighted individual may engage in attacks that a myopic individual would not take. This follows because patient individuals fear giving the enemy the opportunity to strike and so, undergo a preemptive strike. Attacks are triggered even if the current expected payoffs of a confrontation are smaller than those of peace.

The organization of the paper is as follows: Section 2 introduces the conflict formally. Results are in section 3. Section 4 discusses possible future work. Section 5 concludes. Proofs are in the appendix.

## 1.1 Relation with the Existing Literature

While a mathematical analysis of ancient Chinese literature may be new, research on the inefficiency of conflict is not. It has been widely studied in the political economy literature. Garfinkel and Skaperdas (2007) is an excellent survey on conflict, but see also Fearon (1995), Powell (1999), Powell (2004), Powell (2006). However, the connection between this literature and this paper is rather tenuous. The motivating questions differ. The political economy literature focuses on the understanding of the existence of inefficient wars and how institutions affect them. We, instead, focus on the fighting strategies themselves (e.g., when to fight) and the relation between patience and aggressiveness. In addition, most of political economy literature uses statics model whereas we consider a dynamic model. There are a few exceptions to the later point though. For example, Powell (1993) and Acemoglu and Robinson (2001) consider dynamic elements in a model fundamentally different from ours.

Our bilateral setting that eventually divides a constant sum can be related to bargaining models. This is a very large literature that we do not survey here. Serrano (2007) provides an excellent survey. Osborne and Rubinstein (1990) and Roth (1985) also provide a detailed analyses. In the work of Abreu and Gul (2000), Compte and Jehiel (2002) players decide when to take action (concede). The war of attrition structure that arises from bargaining models is orthogonal to our model.

There exists a vast literature interested in mathematical modelling of

military matters, but the examined questions we know of are unrelated to the ones in this paper. A notable pioneering work is Schelling (1980) which developed many insights for rational modelling of armed conflict. O’Neill (1994) provides an extensive survey on the literature.

Our game of warfare is modelled as a stopping game. Technically, stopping games (see Dynkin (1969), Neveu (1975), Yasuda (1985), Rosenberg, Solan, and Vieille (2001), Szajowski (1993), Shmaya and Solan (2004), Ekstrom and Villeneuve (2006), Ohtsubo (1987)) are *reducible* (when players’ actions may change the game permanently) stochastic games. In many reducible games, it is difficult to obtain much more than existence of equilibria. Here we rely on the techniques in Quah and Strulovici (2013) to obtain comparative statics results about best responses.

## 2 Basic Model and Notation

There are two sides 1 and 2. At each period either side either starts a fight or not. If neither side engages in a fight, they get 0 payoffs that period and the game continues into the next period. If one side decides to start a fight, the opponent cannot avoid the dispute and the game ends with one side defeated. The winner gets utility of  $v > 0$ , and the loser utility of  $-l$ ,  $l > v > 0$ . So, confrontations are destructive: the payoff of victory ( $v$ ) is less than the disutility of defeat ( $l$ ).<sup>2</sup> In case of a confrontation at period  $t$ , side 1 is the victor with probability  $p_t$ . At the beginning of each period, both sides observe 1’s probability of winning (hence 2’s probability of winning). Thus, the choice to start a confrontation occurs after observing the odds of victory. The probability  $p_t$  of side 1 winning the confrontation is produced by an independently and identically distributed process with a continuous probability density. The distribution of  $p_t$  is commonly known. We focus on a decision theoretic analysis of the best responses of side 1, against different fixed behaviors of side 2. Side 1 discounts future payoffs with a discount factor  $\beta \in (0, 1)$ .

The conflict starts in the beginning of the process. The key question is when and whether one side will strategically escalate the conflict in an irreversible payoff-relevant move. The critical aspect in our model is the timing of this irreversible escalation which we refer to as the confrontation. In the case of a troubled marriage this may occur when one of the parties files for

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<sup>2</sup>Otherwise, the conflict starts in the first period.

divorce and dispute, say, child-custody. A similar case occurs in commercial litigation, where the evidence/contractual claims can be changing before it goes to court. Here, going to actual trial is to start the confrontation. In many, but certainly not all, cases there is a natural point where the confrontation starts. While we may stay close to the ancient text and describe the conflict in military terms, our main motivation is in ordinary conflicts.

Sun Tzu describes a series of strategic maneuvers designed to confound the adversary and induce the enemy to make a critical error before the actual confrontation. We do not model this part of his work. Instead, we assume that both sides expend an exogenously given effort prior to the confrontation in order to induce the enemy to make errors and avoid their own (and, in this sense, an early preemptive strike is a way to prevent their own mistakes). The results of these efforts and the moves of nature deliver the process of changing odds of victory that we describe exogenously. Thus, our modelling choices focuses on the critical question of the timing of the confrontation.

This section continues as follows: Section 2.1 formally describes the strategies and gives a brief overview of equilibria in a game theoretic formulation. Section 2.2 formally describes aggressiveness. Section 2.3 expands on the meaning of patience in the context of conflicts.

## 2.1 Strategies

### 2.1.1 Basic Notions and Definitions

A history at period  $t$ ,  $h_t = (p_0, p_2, \dots, p_t)$ , is the sequence of probabilities of winning for side 1 up to period  $t$ . Given that the process ends if one side starts the confrontation, we implicitly assume that no side has started a confrontation until period  $t$  while considering a history at period  $t$ . The set of all histories at periods  $t, t + 1, \dots$  generate a growing sequence of  $\sigma$ -algebras,  $\sigma(h_t)$  for the process  $\{p_t\}$  or equivalently a filtration for  $\{p_t\}$ , denoted by  $\{\mathcal{F}_t\}$ . The probability triple (i.e., the filtered probability space) is given by  $(\Omega, \{\mathcal{F}_t\}, P)$ , where  $\Omega$  is the set of all histories of infinite length, i.e.,  $\Omega = [0, 1]^\infty$ , and  $P$  is the probability measure over  $\Omega$  and  $\{\mathcal{F}_t\}$  is the natural filtration. Let  $\mathbb{E}$  be the expectation operator associated with  $P$ .

A pure strategy takes histories as input and returns, as output, the choice of whether to start a confrontation. We formalize pure strategies (in a way that is common in the literature of stopping games) as follows:

**Definition 1.** A pure strategy is a stopping time  $\tau$  for the filtration  $\{\mathcal{F}_t\}$ .<sup>3</sup>

So, a pure strategy determines when to start the confrontation, depending on the current and (perhaps) past odds of victory. Given a history at period  $t$ , a side engages in a confrontation at this history if and only if  $\tau = t$ . For example, consider the *hitting strategy*  $\tau_1 = \inf\{t \geq 0 \mid p_t \geq \bar{p}\}$ . In this strategy, side 1 starts the confrontation when the current odds of victory are at least  $\bar{p}$ . Let  $\tau_i$  be side  $i$ 's pure strategy,  $i = 1, 2$ .

Given a pure strategy profile  $\tau = (\tau_1, \tau_2)$ , the process ends at  $\tau_1 \wedge \tau_2 = \min(\tau_1, \tau_2)$ . Remembering that  $p_t v + (1 - p_t)(-l) = p_t(v + l) - l$ , the overall payoff to side 1 is given by

$$U_1^\tau = \sum_{t=0}^{\infty} \beta^t \mathbb{P}(\tau_1 \wedge \tau_2 = t) \mathbb{E}(p_t(v + l) - l \mid \tau_1 \wedge \tau_2 = t),$$

The payoff can also be written in a more compact form as follows:

$$U_1^\tau = \mathbb{E}(\beta^{\tau_1 \wedge \tau_2} (p_{\tau_1 \wedge \tau_2} (v + l) - l)),$$

**Definition 2.** Given strategy  $\bar{\tau}_2$ , strategy  $\bar{\tau}_1$  is optimal if

$$U_1^{\bar{\tau}_1} = \sup_{\tau_1} \mathbb{E}(\beta^{\tau_1 \wedge \bar{\tau}_2} (p_{\tau_1 \wedge \bar{\tau}_2} (v + l) - l))$$

That is,  $\bar{\tau}_1$  is optimal if it produces maximal utility, given the strategy of the other side. While the concept of optimality does not appear in Art of War, Sun Tzu does comment on the need for careful contingent planning: “The victorious army first realizes the conditions for victory, and then seeks to engage in battle. The vanquished army fights first, and then seeks victory.”

### 2.1.2 Equilibrium and Hitting Strategies

Our analysis is decision-theoretic, and in that regard we will restrict attention to the best response of side 1, while assuming the behavior of side 2. We assume that side 2 uses a hitting strategy with a threshold  $\bar{p}_2$  lower than 0.5 (i.e.,  $\tau_2 = \inf\{t \geq 0 \mid p_t \leq \bar{p}_2\}$ ). This rules out the case where side 2 is so aggressive and attacks with greater chance of losing than winning.

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<sup>3</sup>Let  $\mathbb{N}$  be the set of natural numbers. Given a triple  $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$ , a stopping time is a random variable  $\tau: \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ , such that the event  $(\tau \leq t)$  is  $\mathcal{F}_t$ -measurable.

The restriction to hitting strategies are motivated by the fact that in the case of an i.i.d. process the key characterization result in Ohtsubo (1987) implies that an equilibrium in hitting strategies exists. In the appendix we show that the results in Ohtsubo (1987) applies to our model (Theorem 1). Moreover, side 1's best response to a hitting strategy is almost surely a hitting strategy and can be characterized explicitly under suitable assumptions (Proposition 2). Hence, hereafter we assume hitting time strategies.

## 2.2 Aggressiveness

Aggressiveness is measured by the tendency to initiate the conflict early on. In this setting, aggressiveness can be defined, in a simple way, using the thresholds  $\bar{p}_1$  and  $\bar{p}_2$ . The lower the threshold  $\bar{p}_1$  the less demanding the odds of victory side 1 requires to start a war. Hence, *the lower the threshold  $\bar{p}_1$  the more aggressive side 1 is*. Analogously, consider 2's hitting strategy of attacking when 1's odds of victory  $p_t$  are below  $\bar{p}_2$  (and, hence, 2's odds of victory  $1 - p_t$  are above  $1 - \bar{p}_2$ ). The higher the threshold  $\bar{p}_2$  the less demanding the odds of victory side 2 requires to start a war. Hence, *the higher the threshold  $\bar{p}_2$  the more aggressive side 2 is*.

## 2.3 The Discount Factor

The common interpretation of the discount rates is time-preference. This is also how we interpret and measure the degree of patience in this model.

The degree of farsightedness is close to the degree of patience. A decision maker who highly, but not completely, discounts the future is not the same as a near myopic decision maker. But these concepts are so closely related that the results are bound to remain qualitatively the same, no matter which modelling choice is made. We opted for a discount factor (as opposed to an exact measure of patience) because it is analytically convenient and it is closer to most of the literature on repeated and stochastic games. Thus, along with the literature, as *the discount factor  $\beta$  gets larger, side 1 gets more patient*.



### 3 Main Results

**Proposition 1.** *Consider side 1’s best response to a fixed hitting strategy of side 2. If the support of  $\{p_t\}$  is included in  $(\frac{l}{l+v}, 1)$  then a more patient side 1 is less aggressive. If the support of  $\{p_t\}$  is included in  $(0, \frac{l}{l+v})$  then a more patient side 1 is more aggressive.<sup>4</sup>*

Proposition 1 is consistent with Sun Tzu’s argument on the value of waiting. The case where  $\{p_t\}$  support is a subset of  $(\frac{l}{l+v}, 1)$  corresponds to Sun Tzu’s condition: side 1 can never be defeated with high probability, that is any confrontation is more beneficial than peace. In a sufficiently strong defensive position, there are strong incentives to wait for a better opportunity to attack. So, a more patient side waits longer to start a confrontation. The second part of Proposition 1 shows the critical role of the proviso “The good fighters of old first put themselves beyond the possibility of defeat” in Sun Tzu’s aphorism. If  $\{p_t\}$  support is a subset of  $(0, \frac{l}{l+v})$  then side 1 is a vulnerable position and, in a confrontation, is defeated with high probability. Then, the relation between patience and aggressiveness is reversed. In a vulnerable condition, it pays to conduce early preemptive strikes. A more patient side attacks sooner.<sup>5</sup>

In proposition 1, only the discount factor of side 1 is changing, the strategy of side 2 is fixed. The thought experiment we are considering is one in which you have two scenarios. In both scenarios the strategy of side 2 is the same but the degree of patience of side 1 is different in each scenario. The result is about the how the level of aggressiveness of side 1 differ in these two scenarios.

Proposition 1 considers an asymmetric case, where the odds of victory always favor one side. Now consider the case where both sides have equal intrinsic strength. In particular, assume that the odds of victory  $p_t$  are produced from an i.i.d. uniform random variable over  $[0, 1]$ . So, no side

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<sup>4</sup>For a fixed strategy of side 2, let  $\bar{p}_1(\beta)$  denote the optimal threshold when the discount factor is  $\beta$ . For  $\hat{\beta} > \beta$ , if the support of  $\{p_t\}$  is included in  $(\frac{l}{l+v}, 1)$  then  $\bar{p}_1(\hat{\beta}) \geq \bar{p}_1(\beta)$ , if the support of  $\{p_t\}$  is included in  $(0, \frac{l}{l+v})$  then  $\bar{p}_1(\hat{\beta}) \leq \bar{p}_1(\beta)$ .

<sup>5</sup>Proposition 1 can be extended to work against general strategies of side 2 and with more general stochastic processes. However, this would require a broader, and notationally more involved notion of aggressiveness, where stopping times are ordered by stochastic dominance. This broader definition of aggressiveness can be found from the authors upon request. The proof of proposition 1, without these assumptions, follows exactly like the proof in this paper.

has an intrinsic advantage over the other. Utilizing the additional structural assumptions, a closed-form solution is possible. The closed-form solution delivers a better sense of how the best strategies vary with the parameters of the model.

**Proposition 2.** *Suppose  $p_t$  is i.i.d. uniform over  $[0, 1]$ . Fix  $\bar{p}_2 \in (0, 0.5)$  and  $\beta \in (0, 1)$ . Side 1's best response is the hitting time  $\bar{\tau}_1 = \inf\{t \geq 0 \mid p_t \geq \bar{p}_1\}$ , where*

$$\bar{p}_1 = \frac{\left( l + v - \sqrt{(l + v)(l + v - 2l\beta + l\beta^2 - v\beta^2 + 2l\bar{p}_2\beta + 2v\bar{p}_2\beta)} + l\bar{p}_2\beta + v\bar{p}_2\beta \right)}{\beta(l + v)}.$$

So, side 1 starts the conflict when the odds of victory are above  $\bar{p}_1$ . The existence of equilibria in hitting times when both sides have the same discount factor was already directly implied by Ohtsubo (1987). Given the closed-form solution of the best response, it is immediately possible to characterize equilibria. However our main focus here is the decision theoretic framework, thus we proceed with the implications of the best responses. The comparative statics results follow directly from the closed-form solution:

**Corollary 1.** *Suppose  $p_t$  is i.i.d. uniform over  $[0, 1]$  and consider the best response of side 1 for a fixed  $\beta \in (0, 1)$ . Side 1 is more aggressive when facing a more aggressive side 2. (i.e.,  $\bar{p}_1$  decreases as  $\bar{p}_2$  increases).<sup>6</sup>*

A unilateral increase in aggression from the opponent reduces the option value of waiting for a better opportunity to strike. Thus, it increases the incentives to strike early. Increased aggressiveness is best answered by increased aggressiveness.

The thought experiment we are considering in corollary 1 is again with two scenarios. Here the discount factor is the same in both scenarios but in scenario 1 side 1 faces a more aggressive side 2, and in scenario 2 side 1 faces a less aggressive side 2. The result is about how the level of aggressiveness of side 1 differ in these two scenarios.

A myopic side attacks if the expected payoff of the confrontation is positive. Therefore,  $\bar{p}_1^m$ , where

$$\bar{p}_1^m(v + l) - l = 0,$$

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<sup>6</sup>Consider two hitting strategies of player 2,  $\bar{p}_2, \bar{p}'_2 \in (0, 0.5)$  with  $\bar{p}_2 < \bar{p}'_2$ . Let  $\bar{p}_1(\bar{p}_2)$  and  $\bar{p}_1(\bar{p}'_2)$  denote the respective optimal thresholds against  $\bar{p}_2$  and  $\bar{p}'_2$ . If  $p_t$  is i.i.d. uniform over  $[0, 1]$ , then  $\bar{p}_1(\bar{p}_2) < \bar{p}_1(\bar{p}'_2)$ .

is the optimal threshold, if side 1 is myopic.

**Corollary 2.** *Suppose  $p_t$  is i.i.d. uniform over  $[0, 1]$ . Fix  $\bar{p}_2 \in (0, 0.5)$  and  $\beta \in (0, 1)$  and consider the best response of side 1. If  $\bar{p}_2 > \frac{l - \sqrt{(l-v)(l+v)}}{(l+v)}$  then  $\bar{p}_1 < \bar{p}_1^m$ . Thus,  $\bar{p}_1(v + l) - l < 0$ .*

If the opponent is sufficiently aggressive (i.e., if  $\bar{p}_2$  is high enough) then, for a non-myopic side, it is optimal to attack even if the odds of victory are low enough so that 1's expected payoff in the confrontation is negative. A myopic side does not attack if the expected value of a confrontation is negative and, hence, less than the payoff of peace (0). Thus, a non-myopic individual engages in a confrontation that a myopic one does not.

Recall that in each period there are three possible outcomes for each side: peace, defeat and victory. Each one has a corresponding payoff 0,  $l$  and  $v$ . Hence even though some of the thresholds only involve  $v$  and  $l$  the payoff of peace 0 is implicit in the formula.

For a non-myopic individual the confrontation, although possibly in the distant future, is inevitable (due to hitting times being almost surely finite) with one side being the victor and the other side the loser. Although temporary peace is attainable, lasting peace is never a resulting payoff. On the other hand for a myopic side no confrontation, i.e. peace is one of the three possible outcomes. This result delivers another sense in which Sun Tzu's proviso is critical for his argument on the value of waiting. If the opponent is very aggressive then the opponent is very likely to attack and, therefore, a secure position was not attained. Then, for a patient side, it pays to be aggressive and strike early preemptively.

The optimal threshold  $\bar{p}_1$  ( $= \bar{p}_1(\beta)$ ) depends on the discount factor  $\beta$ . The next result now shows that whether the optimal threshold increases or decreases with the discount factor depends on how aggressive the opponent is. Moreover, the relationship between patience and aggressiveness can be monotonic.

**Corollary 3.** *Suppose  $p_t$  is i.i.d. uniform over  $[0, 1]$ . Fix  $\bar{p}_2 \in (0, 0.5)$ . If  $\bar{p}_2 < \frac{l - \sqrt{(l-v)(l+v)}}{(l+v)}$  then a more patient side 1 is less aggressive. If  $\bar{p}_2 > \frac{l - \sqrt{(l-v)(l+v)}}{(l+v)}$  then a more patient side 1 is more aggressive.*

The option value of waiting depends on how aggressive the opponent is. Hence, the results of increased patience depends on how aggressive the

opponent is. If the opponent is not aggressive (i.e.,  $\bar{p}_2$  is below a threshold) then the option value of waiting increases. A more patient side 1 becomes less aggressive. If the opponent is aggressive then the option value of waiting decreases. A more patient side 1 becomes more aggressive.

Corollary 3 shows an alternative way in which Sun Tzu's main point can be formalized. As mentioned, if the opponent is not aggressive (i.e., if  $\bar{p}_2$  is low) then a secure position was obtained. Then, as Sun Tzu's points out, it pays to wait and a more patient individual waits longer before striking. However, this is only so under Sun Tzu's condition. If the opponent is aggressive (i.e., if  $\bar{p}_2$  is high) then a secure position was not obtained. Then, the relationship between patience and aggressiveness is reversed. More patient individuals do not wait and strike sooner preemptively.

## 4 Future Work

Our basic model is in a decision theoretic framework of conflict of one side against a behavioral opponent. The main results are comparative statics on a stopping problem. This is technically challenging. We could successfully obtain results in the cases of very lopsided conflicts or fair ones. Hopefully future research will be able to generalize our results to a broader spectrum of cases. In particular conflicts that are lopsided but are not extremely so are still outside the scope of this paper. This is an open and, we believe, hard problem.

The closed-form solutions in our paper makes it tempting to consider an equilibrium analysis (see our companion paper (Sandroni and Urgan (2015)) for an equilibrium analysis in stopping games, unrelated to eastern philosophy). However, Art of War is written as a practical guide, which is often related to how to best respond to a given situation. This is more naturally captured in a decision theoretic model. In particular, its only in a decision theoretic framework that the level of the aggressiveness of the opponent can be given exogenously and hence its possible to do comparative statics with it.

Our basic model constitutes an application of stopping games to the problem of conflict. Stopping games, by virtue of being *reducible* exhibit very different characteristics from their repeated or irreducible counterparts, and we hope that our model can be a stepping stone for explaining other economic phenomena that are better captured as stopping games.

## 5 Conclusion

This paper is based on a single aphorism in Sun Tzu’s Art of War “The good fighters of old first put themselves beyond the possibility of defeat, and then waited for an opportunity of defeating the enemy.” A formal analysis of this aphorism examines the basic trade-off between obtaining an option of value and giving the opponent an option value. This requires the use of fairly modern techniques.

Art of War is two millennia old. It is riveting how such an ancient text can inspire people and research in modern times. We hope that this paper will be just a first step towards an understanding of Art of War and eastern philosophy in general, using mathematical models.

## 6 Appendix

In the main text we use the term “side” to describe the individuals in the conflict. This emphasizes that our results are in decision-theoretic setting. However, in the appendix we use the term “player” instead of “side” because it makes the mathematical analysis more familiar (to us).

### 6.1 Sufficiency of Pure Strategies in Best Responses and Hitting Strategies

Given a finite history  $h_t = (p_0, p_1, p_2, \dots, p_t)$  and a profile of strategies  $(\tau_1, \tau_2)$ , Noting down that the natural filtration  $\mathcal{F}_t$  by definition includes all the events up to period  $t$ , and in particular  $h_t$ , let  $V_1^\tau(h_t) = (U_1^\tau | \mathcal{F}_t)$  be the continuation payoff for player 1. Recall that  $P(\tau_2 = t)$  is the probability that player 2 starts a confrontation at period  $t$  and similarly  $P(\tau_1 = t)$  is the probability that player 1 starts a confrontation at period  $t$ .

We adopt the notion of a randomized strategy from Yasuda (1985). The equivalence of this notion with others can be found in Solan, Tsirelson, and Vieille (2012).

**Definition 3.** *Let  $\alpha_n$  be a process adapted to  $\mathcal{F}_n$ , with  $0 \leq \alpha_n \leq 1$  for all  $n$ . Let  $x_n$  be a sequence of i.i.d. random variables distributed uniformly*

over  $[0, 1]$  and independently of  $\mathcal{F}_n$ . Then a randomized stopping time for a strategy  $\alpha$  is defined by

$$\tau(\alpha) = \inf\{n \geq 0 : x_n \leq \alpha_n\}$$

If  $\alpha_n$  is equal to 0 or 1 for all  $n$  then the stopping time is pure.

The notable difference of having randomized strategies is that now  $P(\tau_i = t | \mathcal{F}_t) \in [0, 1]$  instead of  $\{0, 1\}$ . Given a (possibly suboptimal) profile of strategies  $(\tau_1, \tau_2)$  the continuation payoff for player 1 is

$$V_1^\tau(h_t) = \sum_{k=t}^{\infty} \beta^{k-t} P(\tau_2 \wedge \tau_1 = k | \mathcal{F}_t) \mathbb{E}(p_k(v+l) - l | \tau_2 \wedge \tau_1 = k, \mathcal{F}_t)$$

Whether players are randomizing or not we have  $-l < \mathbb{E}(p_k(v+l) - l) < v$ , clearly  $P(\tau_2 \wedge \tau_1 = k) \in [0, 1]$  for all  $k$  and  $p_t \in [0, 1]$  for all  $t$  and  $\beta < 1$ . This implies that

$$-l \leq V_1^\tau(h_t) \leq v \text{ for all } h_t.$$

Since the continuation payoffs are bounded, taking a dynamic programming approach akin to Abreu, Pearce, and Stacchetti (1990), we can write the time  $t$  problem of player 1 in the following manner.

$$\begin{aligned} V_1^\tau(h_t) = & P(\tau_2 = t | \mathcal{F}_t)(p_t v - (1 - p_t)l) + \\ & (1 - P(\tau_2 = t | \mathcal{F}_t)) \max_{\alpha \in [0, 1]} [(1 - \alpha)\beta E(V_1^\tau(h_{t+1}) | \mathcal{F}_t) + \alpha(p_t v - (1 - p_t)l)]. \end{aligned} \quad (1)$$

In problem 1, we allow player 1 to start a conflict with probability  $\alpha$ . Notice that the first term is independent of the choice of player 1. Since  $\beta E(V_1^\tau(h_{t+1}) | \mathcal{F}_t)$  is bounded above and below by respectively  $\beta v$  and  $-\beta l$  there exists a unique  $\bar{p}_{1,t}$  such that  $\beta E_t(V_1^\tau(h_{t+1})) = \bar{p}_{1,t}v - (1 - \bar{p}_{1,t})l$ .

Thus, in problem 1 we have

$$\arg \max_{\alpha \in [0, 1]} [(1 - \alpha)\beta E(V_1^\tau(h_{t+1}) | \mathcal{F}_t) + \alpha(p_t v - (1 - p_t)l)] = \begin{cases} 1 & \text{if } p_t > \bar{p}_{1,t} \\ 0 & \text{if } p_t < \bar{p}_{1,t} \\ [0, 1] & \text{if } p_t = \bar{p}_{1,t} \end{cases}$$

Equivalently, for  $\tau_1$  that is a best response, we have

$$\begin{aligned} P(\tau_1 = t | \mathcal{F}_t) &= 1 \text{ if } p_t > \bar{p}_{1,t} \\ P(\tau_1 = t | \mathcal{F}_t) &= 0 \text{ if } p_t < \bar{p}_{1,t} \\ P(\tau_1 = t | \mathcal{F}_t) &\in [0, 1] \text{ if } p_t = \bar{p}_{1,t} \end{aligned} \quad (2)$$

**Observation 1.** *We assume  $\{p_t\}$  to have a continuous pdf at every  $t$ . Thus, almost surely,  $p_t \neq \bar{p}_{1,t}$  for all  $t$ .*

Thus, due to observation 1 for almost surely for all  $t$ ,  $P(\tau_1 = t | \mathcal{F}_t) \in \{0, 1\}$  which means the strategy is pure.

The following two lemmas formalize the intuitive notion that optimally stopping an i.i.d. process which stops with a fixed probability every period (via the fixed threshold strategy of side 2) can be done by a hitting time strategy. Other than formalizing this notion, the lemma's are unrelated to the rest of the proofs and can be skipped.

**Lemma 1.** *Suppose the process generating the odds is i.i.d.. If player 2 employs a fixed threshold strategy, then player 1's best response is almost surely a stationary strategy, i.e. it is constant across time.*

*Proof.* Let  $\bar{p}_2$  denote a fixed threshold strategy for player 2, i.e. player 2 starts a confrontation whenever  $p_t \leq \bar{p}_2$ . From observation 1 and equation 2 we know that we can restrict attention to pure strategies for player 1 without loss of generality. Thus, player 1 selects a pure strategy  $\tau_1$  in order to maximize the following:

$$\begin{aligned} \max_{\tau_1} \sum_{t=0}^{\infty} \beta^t [P(\tau_1 = t)(\mathbb{E}(p_t(v+l) - l | \tau_1 = t)(1 - F(\bar{p}_2)) \\ + F(\bar{p}_2)\mathbb{E}(p_k(v+l) - l | p_t < \bar{p}_2)] \end{aligned}$$

Where  $F(\bar{p}_2) = P(p_t \leq \bar{p}_2)$ . Since  $\tau_1$  is pure, the action space for each period is finite. Thus, by (Blackwell 1965) (7b) we know that the optimal strategy  $\tau_1^*$  is Markovian. However since player 2 employs a fixed threshold strategy, Markovian behavior only depends on the current realization, i.e.  $\exists C_t \subseteq [0, 1]$  such that  $p_t \in C_t \iff \tau_1^* = t$ . Moreover since the process is i.i.d. Markovian behavior is almost surely stationary, i.e.,  $C_t = C_s$  for all  $t$  and  $s$  almost surely. Thus for some  $C \subseteq [0, 1]$ ,  $C_t = C$  for all  $t$  almost surely.  $\square$

**Lemma 2.** *Suppose the process generating the odds is i.i.d.. If player 2 employs a fixed threshold strategy, then player 1's best response is almost surely a fixed threshold strategy.*

*Proof.* From lemma 1 we know that the best response to a fixed stationary strategy is characterized by a confrontation region  $C \subseteq [0, 1]$ , that is constant across time, i.e.  $\tau_1^* = t \iff p_t \in C$ . Let  $V_1^{\tau_1^*, \bar{p}_2}(h_t)$  denote the continuation payoff for player 1 when player 1 employs the optimal strategy  $\tau_1^*$  and player 2 employs a fixed threshold strategy with threshold  $\bar{p}_2$ . Since both players employ strategies that are constant across time, it must be the case that for any period  $t$ ,  $\mathbb{E}(V_1^{\tau_1^*, \bar{p}_2}(h_{t+1})|\mathcal{F}_t)$  is constant. That is  $\exists v_1(\tau_1^*, \bar{p}_2) \in [-l, v]$  such that  $\mathbb{E}(V_1^{\tau_1^*, \bar{p}_2}(h_{t+1})|\mathcal{F}_t) = v_1(\tau_1^*, \bar{p}_2)$  for all  $t$ . But then for any period  $t$  the problem of player 1 is as follows:

$$P(\tau_2 = t)(p_t v - (1 - p_t)l) + (1 - P(\tau_2 = t)) \max_{\alpha \in [0, 1]} [(1 - \alpha)\beta v_1(\tau_1^*, \bar{p}_2) + \alpha(p_t v - (1 - p_t)l)].$$

Since  $v_1(\tau_1^*, \bar{p}_2)$  is bounded above and below by respectively  $\beta v$  and  $-\beta l$  there exists a unique  $\bar{p}_1$  such that  $\beta E_t(V_1^{\tau_1^*}(h_{t+1})) = \bar{p}_1 v - (1 - \bar{p}_1)l$ . Thus

$$\arg \max_{\alpha \in [0, 1]} [(1 - \alpha)\beta v_1(\tau_1^*, \bar{p}_2) + \alpha(p_t v - (1 - p_t)l)] = \begin{cases} 1 & \text{if } p_t > \bar{p}_1 \\ 0 & \text{if } p_t < \bar{p}_1 \\ [0, 1] & \text{if } p_t = \bar{p}_1 \end{cases}$$

But then  $\tau_1^* = t \iff p_t > \bar{p}_1$  thus the best response to a fixed threshold strategy is a fixed threshold strategy, with a measure 0 indifference in  $\bar{p}_1$ , which we break in favor of a confrontation.  $\square$

We present a slightly modified version of a result of Quah and Strulovici (2013) without proof.

**Lemma 3** (Quah & Strulovici). *Let  $H$  be a regular stochastic process of bounded variation such that  $E[H_t] \leq E[H_{\bar{t}}]$  for all  $t \in [0, \bar{t})$ , and let  $\gamma$  be a positive regular deterministic process. Then,*

$$E \left[ \int_0^{\bar{t}} \gamma_s dH_s \right] \geq \gamma(0)E[H(\bar{t}) - H(0)].^7$$

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<sup>7</sup>The proof is identical to theirs, the change in assumptions only allow us to use integration by parts without assuming  $\gamma$  is increasing.



### 6.1.1 Proof of Proposition 1

Here we will show one side of the argument, the other side is analogous. Suppose that the support of  $p_t$  is  $(\frac{l}{l+v}, 1)$ .

Let  $w_{1,t} = p_t(v + l) - l$  denote the immediate expected returns of player 1 after  $p_t$  is realized.

Since the support is  $(\frac{l}{l+v}, 1)$  necessarily for any realization of  $p_t$  it must be the case that  $w_{1,t} \geq 0$ . For a strategy profile  $\tau(\beta) = (\tau_1(\beta), \tau_2(\beta))$ , let  $V_1^{\tau(\beta)}(h_k)$  denote the continuation payoff of player 1 starting from history  $h_k$ . Then using a Lebesgue-Stieltjes differential form we must have

$$V_1^{\tau(\beta)}(h_k) = \mathbb{E} \left[ \sum_{t=k}^{\tau_1(\beta) \wedge \tau_2(\beta)} \beta^{t-k} \Delta w_{1,t-1} + w_{1,t}(\Delta \beta^{t-k-1}) \right] \geq 0$$

Where  $\Delta$  denotes the forward difference operator, i.e.  $\Delta w_{1,t-1} = w_{1,t} - w_{1,t-1}$  and  $\Delta \beta^{t-k-1} = \beta^{t-k} - \beta^{t-k-1}$ .

Since we know that at the optimal hitting threshold player 1's immediate payoff has to be equal to the continuation utility we must identify the change in the continuation utility in order to identify the change in the threshold. First, keeping the strategy of the second player constant, for  $\hat{\beta} > \beta$  utilizing lemma 3 we have

$$\mathbb{E} \left[ \sum_{t=k}^{\tau_1(\beta) \wedge \tau_2(\beta)} \hat{\beta}^{t-k} \Delta w_{1,t-1} + w_{1,t}(\Delta \hat{\beta}^{t-k-1}) \right] \geq \frac{\hat{\beta}}{\beta} \mathbb{E} \left[ \sum_{t=k}^{\tau_1(\beta) \wedge \tau_2(\beta)} \beta^{t-k} \Delta w_{1,t-1} + w_{1,t}(\Delta \beta^{t-k-1}) \right] \geq 0$$

Since  $w_{1,t} \geq 0$  for all realizations, replacing  $\Delta \beta^{t-k}$  with  $\Delta \hat{\beta}^{t-k}$  on the left hand side, and dropping  $\frac{\hat{\beta}}{\beta} \geq 1$  yields

$$\mathbb{E} \left[ \sum_{t=k}^{\tau_1(\beta) \wedge \tau_2(\beta)} \hat{\beta}^{t-k} \Delta w_{1,t-1} + w_{1,t}(\Delta \hat{\beta}^{t-k-1}) \right] \geq \mathbb{E} \left[ \sum_{t=k}^{\tau_1(\beta) \wedge \tau_2(\beta)} \beta^{t-k} \Delta w_{1,t-1} + w_{1,t}(\Delta \beta^{t-k-1}) \right]$$

Hence, the continuation values increase when the strategies remain the same. Now, we need to show that the optimal stopping rules, will indeed be ordered. Letting  $\tau_i(\hat{\beta})$  denote the optimal strategy with  $\hat{\beta}$ , for a contradiction

suppose  $\Psi = \{\omega : \tau_1(\beta) \geq \tau_1(\hat{\beta})\}$  has strictly positive probability, but then by a similar calculation to above on  $\Psi$  we will have

$$\mathbb{E} \left[ \sum_{\tau_1(\hat{\beta}) \wedge \tau_2(\beta)}^{\tau_1(\beta) \wedge \tau_2(\beta)} \hat{\beta}^{t-k} \Delta w_{1,t-1} + w_{1,t}(\Delta \hat{\beta}^{t-k-1}) | \Psi \right] \geq \mathbb{E} \left[ \sum_{\tau_1(\hat{\beta}) \wedge \tau_2(\beta)}^{\tau_1(\beta) \wedge \tau_2(\beta)} \beta^{t-k} \Delta w_{1,t-1} + w_{1,t}(\Delta \beta^{t-k-1}) | \Psi \right]$$

But then waiting is better even on the set  $\Psi$ , contradicting its optimality. Thus we must have the stopping times weakly ordered.

### 6.1.2 Proof of Proposition 2

Let  $\bar{p}_2$  denote a fixed threshold for player 2. From lemma 6.1 we know that player 1 will optimally utilize a fixed threshold strategy against  $\bar{p}_2$ . Let  $p_1 > \bar{p}_2$  denote a fixed threshold for player 1. Then, with a slight abuse of notation the continuation payoff of player 1 denoted,  $E(V_1 | \bar{p}_2, p_1)$ , can be written down as follows:

$$E(V_1 | \bar{p}_2, p_1) = \sum_{t=0}^{\infty} \beta^t (F(p_1) - F(\bar{p}_2))^t \left( \int_0^{\bar{p}_2} [p(v+l) - l] f(p) dp + \int_{p_1}^1 [p(v+l) - l] f(p) dp \right)$$

Plugging in the uniform distribution yields the following expression:

$$E(V_1 | \bar{p}_2, p_1) = \sum_{t=0}^{\infty} \beta^t (p_1 - \bar{p}_2)^t \left( \int_0^{\bar{p}_2} [p(v+l) - l] dp + \int_{p_1}^1 [p(v+l) - l] dp \right)$$

With some algebra, the expression simplifies to:

$$E(V_1 | \bar{p}_2, p_1) = \frac{\frac{1}{2}v - \frac{1}{2}l + lp_1 - l\bar{p}_2 - \frac{1}{2}lp_1^2 + \frac{1}{2}l\bar{p}_2^2 - \frac{1}{2}vp_1^2 + \frac{1}{2}v\bar{p}_2^2}{1 - \beta(p_1 - \bar{p}_2)} \quad (3)$$

Due to equation 2 we know that if  $p_1$  is selected optimally, at the exact threshold (i.e.  $p_t = p_1$ ) the discounted continuation payoff and the payoff from immediate confrontation should be the same.

Thus letting  $\bar{p}_1$  denote the optimal threshold we must have

$$\beta \left( \frac{\frac{1}{2}v - \frac{1}{2}l + l\bar{p}_1 - l\bar{p}_2 - \frac{1}{2}l\bar{p}_1^2 + \frac{1}{2}l\bar{p}_2^2 - \frac{1}{2}v\bar{p}_1^2 + \frac{1}{2}v\bar{p}_2^2}{1 - \beta(\bar{p}_1 - \bar{p}_2)} \right) = \bar{p}_1 v - (1 - \bar{p}_1)l$$

$$\bar{p}_1 = \frac{\left( l + v - \sqrt{(l+v)(l+v - 2l\beta + l\beta^2 - v\beta^2 + 2l\bar{p}_2\beta + 2v\bar{p}_2\beta)} + l\bar{p}_2\beta + v\bar{p}_2\beta \right)}{\beta(l+v)} \quad (4)$$

Given that  $\bar{p}_2 \leq 1/2$  we have  $\bar{p}_1 \geq 1/2$  since

$$\bar{p}_1 \geq \frac{1}{2} + \frac{1}{\beta} \left( 1 - \frac{\sqrt{(l+v)(l+v - (l-v)(\beta - \beta^2))}}{l+v} \right)$$

Furthermore given that  $\bar{p}_2 \geq 0$  we have  $\bar{p}_1 \leq 1$  since

$$\bar{p}_1 \leq \frac{l+v - \sqrt{(l+v)(l+v - 2l\beta + l\beta^2 - v\beta^2)}}{\beta(l+v)} \leq 1$$

### 6.1.3 Proof of Corollary 1

Differentiating equation 4 with respect to  $\bar{p}_2$ ,

$$\frac{\partial}{\partial \bar{p}_2} \left( \frac{\left( l + v - \sqrt{(l+v)(l+v - 2l\beta + l\beta^2 - v\beta^2 + 2l\bar{p}_2\beta + 2v\bar{p}_2\beta)} + l\bar{p}_2\beta + v\bar{p}_2\beta \right)}{\beta(l+v)} \right)$$

$$= 1 - \frac{\left( \sqrt{(l+v)(l+v - 2l\beta + l\beta^2 - v\beta^2 + 2l\beta\bar{p}_2 + 2v\beta\bar{p}_2)} \right)}{l+v - 2l\beta + l\beta^2 - v\beta^2 + 2l\beta\bar{p}_2 + 2v\beta\bar{p}_2} \leq 0$$

The term above is negative because  $\bar{p}_2 \leq 1/2$

### 6.1.4 Proof of Corollary 2

Using equation 4 we solve  $\bar{p}_1(v+l) - l < 0$  for  $\bar{p}_2$ . Straightforward algebra yields

$$\begin{aligned}
0 &> \frac{\left( (l+v) - \sqrt{(l+v)(l+v-2l\beta+l\beta^2-v\beta^2+2l\bar{p}_2\beta+2v\bar{p}_2\beta)} + l\bar{p}_2\beta + v\bar{p}_2\beta \right)}{\beta(l+v)}(v+l) - l \\
&\sqrt{(l+v)(l+v-2l\beta+l\beta^2-v\beta^2+2l\bar{p}_2\beta+2v\bar{p}_2\beta)} > (l+v)(1-\beta\bar{p}_2) - l\beta \\
\bar{p}_2 &> \frac{l^2 + lv - \sqrt{(l-v)(l+v)^3}}{(l+v)^2} \\
\bar{p}_2 &> \frac{l - \sqrt{(l-v)(l+v)}}{(l+v)}
\end{aligned}$$

### 6.1.5 Proof of Corollary 3

Suppose that the second player is decision theoretic player that is playing with a stationary hitting time, characterized by the threshold  $\bar{p}_2$ . Since the discount factor has no impact on the payoff of immediate confrontation using problem 1 and equation 2 it is easy to see the sign of the continuation value determines the comparative statics for the single player problem. Focusing on equation 3 we simplify the necessary condition for the continuation value to be positive ,

$$\frac{\frac{1}{2}v - \frac{1}{2}l + lp_1 - l\bar{p}_2 - \frac{1}{2}lp_1^2 + \frac{1}{2}l\bar{p}_2^2 - \frac{1}{2}vp_1^2 + \frac{1}{2}v\bar{p}_2^2}{1 - \beta(p_1 - \bar{p}_2)} \geq 0$$

Plugging in the optimal value of  $\bar{p}_1$  and isolating  $\bar{p}_2$  we have

$$\begin{aligned}
\bar{p}_2 &< \frac{l^2 + lv - \sqrt{(l-v)(l+v)^3}}{(l+v)^2} \\
\bar{p}_2 &< \frac{l - \sqrt{(l-v)(l+v)}}{(l+v)}
\end{aligned}$$

## 6.2 Equilibrium Existence

The expectation of a resulting conflicting given current odds is crucial to identify the strategies, as well as characterizing the equilibria. In that direction

for each  $m \in \mathbb{N}$  let the following sequences of random variables  $\{(\psi_n^m, \phi_n^m)\}_{n=0}^m$  be defined by backward induction in the following manner:

$$\begin{aligned} (\psi_m^m, \phi_m^m) &= (p_m(v+l) - l, v - p_m(v+l)) \\ (\psi_n^m, \phi_n^m) &= \begin{cases} (p_n(v+l), v - p_n(v+l)) & \text{if } (p_n(v+l), w_n^2) \geq (\beta E(\psi_{n+1}^m | \mathcal{F}_n), \beta E(\phi_{n+1}^m | \mathcal{F}_n)) \\ (\beta E(\psi_{n+1}^m | \mathcal{F}_n), \beta E(\phi_{n+1}^m | \mathcal{F}_n)) & \text{o/w} \end{cases} \end{aligned}$$

Here we first notice that this construction is very similar to the construction of a Snell envelope, extended to accommodate 2 players. (Ohtsubo 1987) shows that  $\psi_n = \lim_{m \rightarrow \infty} \psi_n^m$  and  $\phi_n = \lim_{m \rightarrow \infty} \phi_n^m$  are well defined. Similar to the Snell envelope, the P-limit of these sequences define the essential suprema, which was shown to be equal to a pair of equilibrium payoffs in the game (Ohtsubo 1987). Notice that in this characterization there are no strategies, hence the information from the filtration  $\mathcal{F}_n$  is limited to just how many future periods are accounted for, thus in the limit  $E(\psi_{k+1} | \mathcal{F}_k)$  and  $E(\phi_{k+1} | \mathcal{F}_k)$  converge to two real numbers  $\psi, \phi \in [-l, v]$ .

Here we present the equilibrium theorem without proof.

**Theorem 1** (Ohtsubo).

$$\bar{\tau}^1 = \inf\{k \geq 0 | \beta E(\psi_{k+1} | \mathcal{F}_k) \leq w_k^1\}$$

$$\bar{\tau}^2 = \inf\{k \geq 0 | \beta E(\phi_{k+1} | \mathcal{F}_k) \leq w_k^2\}$$

constitute an equilibrium of this game. Where, the sequences  $\psi, \phi$  correspond to the equilibrium values.

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